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Duality and universality in non-equilibrium lattice models

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Abstract. We introduce two kinds of discrete-time non-equilibrium lattice models (stochastic cellular automata), which we call the *creation process* η_t and the *branching process* $\tilde{\eta}_t$. In special cases the former process can be identified with the site or bond directed percolation models. When the system is defined on a d-dimensional finite lattice with size L, these processes are determined by $2^{L^d} \times 2^{L^d}$ transition matrices, M_L and $\tilde{M}_L$, respectively. It is proved that subject to certain relations between the parameters of these models, M_L and the transpose of $\tilde{M}_L$ are conjugate and thus the characteristic polynomials become equal to each other, $\det(M_L - \lambda E) = \det(\tilde{M}_L - \lambda E)$, for arbitrary $L \geqslant n$, where E is the identity matrix. Since dynamical critical exponents as well as critical values will be determined by the asymptotic behaviour in the limit $L \to \infty$ of the large eigenvalues of the transition matrix, our result implies that, if continuous phase transitions and critical phenomena are observed, these two processes belong to the same universality class. In proving the equality, we use the relation which is called the *coalescing duality* in probability theory.

1. Introduction

Phase transitions occur in a wide variety of non-equilibrium lattice models such as the basic contact process [1–3] and its modified or generalized processes [4–13], branching annihilating random walks [14–17], multi-particle annihilation models [18–20, 6] and the ZGB model [21]. When the transition is continuous, critical phenomena are observed and universality classes of non-equilibrium lattice models have been studied. Numerical studies by Monte Carlo simulations [22] and series expansions [23] show that the critical phenomena observed in the d-dimensional contact-process-type models belong to the same universality class with the $(d + 1)$-dimensional directed percolation (in short DP). Field-theoretical arguments leads to the conclusion that the DP and the Reggeon field theory belong to the same universality class [24]. This DP universality class is very wide and the following statement is proposed for {0, 1}-valued models by Janssen [25] and Grassberger [26], which is called the DP *conjecture* in Dickman [27]: if a model with a scalar order parameter exhibits a continuous transition into a unique absorbing state, the critical behaviour is generically of the directed percolation type. Although there is no counterexample to this conjecture, its proof is still an open problem. We will observe more complicated phenomena if the assumption of the DP conjecture is not satisfied [28–30] or the system has additional conservation laws [15]. So far there has been no general criterion for the universality classification of non-equilibrium lattice models.

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Consider a discrete-time version of the {0, 1}-valued non-equilibrium lattice model with an absorbing state. When the system is defined on a d-dimensional finite lattice with size L, the dynamics is described by a $2^{L^d} \times 2^{L^d}$ transition matrix M_L. If we use the rule in which each site is updated simultaneously, the system is also called *stochastic cellular automata* [31]. Let $\lambda_0, \lambda_1, \lambda_2, \ldots$ be the eigenvalues of M_L. Since an absorbing state is a trivial stationary state, it gives a left eigenvector of M_L corresponding to the largest eigenvalue $\lambda_0 \equiv 1$. Other eigenvalues depend on the parameters of model. For simplicity, here we consider a one-parameter model with p denoting a parameter. We assume that there is a continuous phase transition at $p = p_c$ in the thermodynamic limit $L \to \infty$; for $p \leqslant p_c$ the stationary state is unique and given by the trivial absorbing state, while for $p > p_c$ there is another active stationary state. As we have seen in many equilibrium lattice models, it is expected that critical exponents as well as the critical value p_c can be determined by the asymptotic behaviour as $L \to \infty$ of the non-trivial eigenvalues $\{\lambda_i\}_{i \geqslant 1}$.

Kinzel [31] discussed phase transitions and critical phenomena of the stochastic cellular automata by analysing the scaling behaviour of the largest non-trivial eigenvalue λ_1. He noticed that the correlation time $\tau(p, L)$ of the system with size L and a parameter p is given as

$$\tau(p, L) = -(\ln \lambda_1(p, L))^{-1} . \tag{1.1}$$

He proposed the following scaling for $\tau(p, L)$, which will hold asymptotically for large L and $\epsilon = p - p_c \to 0$:

$$\tau\left(\epsilon, \frac{1}{L}\right) = b^z \tau\left(b^{1/\nu_\perp}\epsilon, \frac{b}{L}\right) \tag{1.2}$$

for any scale factor b. Here $z = \nu_\parallel/\nu_\perp$ is the dynamical critical exponent. He calculated the following quantity for large L:

$$Y_L \equiv \ln[\tau(p, L)/\tau(p, L-1)]/\ln[L/(L-1)] \tag{1.3}$$

and evaluated p_c and z by the following asymptote for $L \to \infty$:

$$\begin{aligned} &Y_L \sim L && p > p_c \\ &Y_L = z && p = p_c \\ &Y_L \to 0 && p < p_c . \end{aligned} \tag{1.4}$$

Recently ben-Avraham *et al* [32] showed that the second non-trivial eigenvalue of the transition matrix λ_2 has its maximum at p_c. Their argument implies that the dynamical exponent z can be obtained by

$$z = -\lim_{L \to \infty} \frac{\ln(1 - \lambda_2(p, L))}{L} . \tag{1.5}$$

Both theories are phenomenologically based on the scaling assumptions and are far from rigorous. However, we believe that p_c and critical exponents are determined by the asymptote of eigenvalues $\{\lambda_i(p, L)\}_{i=1,2,3,\ldots}$ in $L \to \infty$ and thus the following statement should be true. Consider two different non-equilibrium lattice models showing critical phenomena, whose transition matrices are given by M_L and $\bar{M}_L$, respectively, for each L. For a finite matrix M, let $\gamma_M(\lambda) = \det(M - \lambda E)$ be the characteristic polynomial, where E is the identity matrix. Then if

$$\gamma_{M_L}(\lambda) = \gamma_{\bar{M}_L}(\lambda) \qquad \text{for any } L \tag{1.6}$$

then these two models belong to the same universality class.

In the present paper we introduce two kinds of discrete-time non-equilibrium lattice models, which we call the *n-range creation process* η_t and the *n-range branching process* $\tilde{\eta}_t$. The former (resp. the latter) has 2^n parameters $\{a_A\}$ (resp. $\{p_A\}$) and is defined by a transition matrix M_L (resp. $\tilde{M}_L$,) for the finite system with size L. Dynamics of these two processes are quite different from each other and $M_L \neq \tilde{M}_L$ in general. However, we will prove for $n \geqslant 2$ that if we put some relations between $\{a_A\}$ and $\{p_A\}$, the equality (1.6) holds. This result implies that the processes η_t and $\tilde{\eta}_t$ belong to the same universality class, if they show critical phenomena. Before starting a general argument, we first give a simple example using the special case of the dimension $d = 1$ and the range $n = 2$ in section 2. In section 3 we define the n-range creation process and the n-range branching process and give the formulae by which we can derive M_L and $\tilde{M}_L$ for any $L \geqslant n$ (lemmas 3.1 and 3.2). We also give the relations between the matrix elements of M_L and $\tilde{M}_L$ (proposition 3.3), which should be called the *duality relation* as explained in remark 3.1. Section 4 is devoted to prove the main theorem (theorem 4.1). We give some remarks in section 5.

2. Demonstration

In order to explain the motivation of present study, we will consider here two processes η_t and $\tilde{\eta}_t$ on a one-dimensional chain of L sites with a periodic boundary condition ($L \geqslant 2$). Since the process η_t can be constructed by using site-directed percolation on the two-dimensional spatio-temporal plane $\{1, 2, \ldots, L\} \times \{0, 1, 2, \ldots\} \ni (x, t)$, we call it the SDP. Let $\{S_{x,t}\}$ be a collection of independent and identically distributed $\{0, 1\}$-valued random variables with an expectation $\langle S_{x,t} \rangle = p$, $0 \leqslant p \leqslant 1$. The process $\eta_t \in \{0, 1\}^{\{1,2,\ldots,L\}}$ is a Markov chain and the state at site x and time $t + 1$ is determined by (see figure 1)

$$\eta_{t+1}(x) = S_{x,t}(\eta_t(x) + \eta_t(x+1) - \eta_t(x)\eta_t(x+1)) . \tag{2.1}$$

It should be noted that if $\eta_0(x) = \delta_{x,1}$ the set of sites $C = \{(x, t) : \eta_t(x) = 1\}$ is the directed percolation cluster starting from $(1, 0)$ on the spatio-temporal plane.

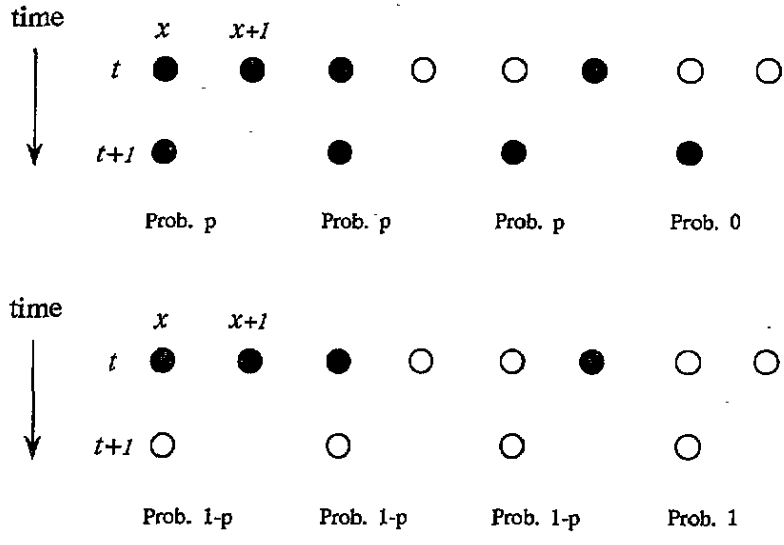

Figure 1. The elementary processes of the SDP.

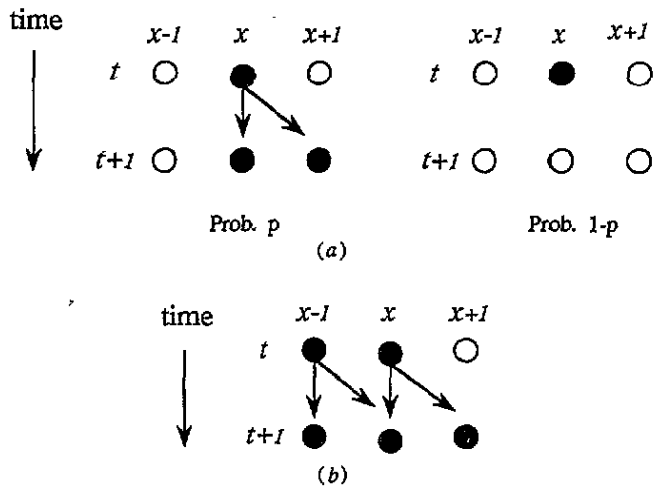

Figure 2. (*a*) The elementary processes of the CBP. Each particle branches with probability p or annihilates with probability $1-p$. (*b*) If two particles land on the same site x, they coalesce into one.

Define the state vector

$$\psi_L(\eta_t) \equiv \text{Prob}(\eta_t = \{\eta_t(x) : x \in \{1, 2, \ldots, L\}\}) . \tag{2.2}$$

Then its transition is described by

$$\psi_L(\eta_{t+1}) = \sum_{\{\eta_t\}} \psi_L(\eta_t) M_L(\eta_t, \eta_{t+1}) \tag{2.3}$$

and the $2^L \times 2^L$ matrix M_L is called the transition matrix. Note that in Kinzel [31] and ben-Avraham *et al.* [32] the transposed matrix of M_L is called the transfer matrix.

In the case of $L = 2$, the transition matrix of SDP is given by

$$\begin{aligned}(M_2)_{i,j} &\equiv \text{Prob}(2\eta_{t+1}(1) + \eta_{t+1}(2) = j | 2\eta_t(1) + \eta_t(2) = i) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ (1-p)^2 & (1-p)\,p & (1-p)\,p & p^2 \\ (1-p)^2 & (1-p)\,p & (1-p)\,p & p^2 \\ (1-p)^2 & (1-p)\,p & (1-p)\,p & p^2 \end{bmatrix} .\end{aligned} \tag{2.4}$$

The other process $\tilde{\eta}_t$ shall be called the coalescing branching process (CBP). This process is regarded as an interacting particle system in the discrete time. If we regard the state $\tilde{\eta}_t(x) = 1$ as the occupation of (x, t) by a particle, the dynamics of particles are defined as follows; each particle located at (x, t) branches to the two sites $(x, t+1)$and $(x+1, t+1)$ with probability p or annihilates with probability $1-p$ (see figure 2(*a*)). The process is coalescing: if two particles land on the same site, they coalesce into one (see figure 2(*b*)). For $L = 2$ the transition matrix of CBP is given by

$$\tilde{M}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-p & 0 & 0 & p \\ 1-p & 0 & 0 & p \\ (1-p)^2 & 0 & 0 & 2p-p^2 \end{bmatrix} . \tag{2.5}$$

Although $M_2 \neq \tilde{M}_2$, we find

$$\det(M_2 - \lambda E) = (2p - p^2)\lambda^2 + (-1 - 2p + p^2)\lambda^3 + \lambda^4$$
$$= \det(\tilde{M}_2 - \lambda E) \tag{2.6}$$

where E is the 4×4 identity matrix.

When $L = 3$, the transition matrices become 8×8. By using *Mathematica*, we find the equality of characteristic polynomials also for this case. Now the problem is to consider the cases $L \geqslant 4$.

3. Transition matrices and duality relation

In this section we consider two kinds of discrete-time interacting particle systems on a finite lattice with periodic boundary conditions, in which each lattice site can be occupied by at most one particle. The former process will be called the *n-range creation process*, since the creation rate of a particle on a site x at the time $t+1$ depends on the particle configuration on n sites (the site x and its neighbours). In the latter process, which we will call the *n-range branching process*, each particle branches and give birth to at most n particles on the neighbouring sites. Consider a d-dimensional finite integer lattice with size L and periodic boundary condition, $\Lambda_L^{(d)} = \{1, 2, \ldots, L\}^d$. The state space is $X_L^{(d)} = \{0, 1\}^{\Lambda_L^{(d)}}$ and the processes are determined by the $2^{L^d} \times 2^{L^d}$ transition matrices.

3.1. n-range creation process η_t

The n-range creation process η_t on $\Lambda_L^{(d)}$ is a discrete-time Markov chain with $t = 0, 1, 2, \ldots$. At each site $x \in \Lambda_L^{(d)}$, the random variable $\eta_t(x)$ takes one of the two possible values; $\eta_t(x) = 1$ and $\eta_t(x) = 0$ representing occupancy or vacancy of a particle, respectively. Let $R_n = \{0, d_1, d_2, \ldots, d_{n-1}\}$ be a set of n sites ($n \leqslant L$). We name R_n the range. We assign a set of 2^n parameters $\{a_A : A \subseteq R_n\}$ with $0 \leqslant a_A \leqslant 1$. Let $R_n^x = R_n + x = \{x, x + d_1, x + d_2, \ldots, x + d_{n-1}\}$ and $A^x = A + x$. We assume the translation invariance of kinetics and define $a_{A^x} = a_A$. The value $\eta_{t+1}(x)$ is determined depending on the configuration $\{\eta_t(s) : s \in R_n^x\}$ by the following stochastic rule (see figure 3(*a*)). For each $x \in \Lambda_L^{(d)}$, if $\eta_t(y) = 1$ for all $y \in A^x$ and $\eta_t(z) = 0$ for all other $z \in R_n^x \setminus A^x$, then

$$\eta_{t+1}(x) = \begin{cases} 1 & \text{with probability} \quad a_A \\ 0 & \text{with probability} \quad 1 - a_A\,. \end{cases} \tag{3.1}$$

For any configuration $\eta_t = \{\eta_t(x) : x \in \Lambda_L^{(d)}\} \in X_L^{(d)}$ we can define a set of sites as

$$A_t = \{x : \eta_t(x) = 1\}\,. \tag{3.2}$$

The process η_t can be identified with the time evolution of the set A_t in $Y_L^{(d)} \equiv$ the collection of subsets of $\Lambda_L^{(d)}$. The $2^{L^d} \times 2^{L^d}$ transition matrix M_L of the d-dimensional n-range creation process is thus defined on $Y_L^{(d)} \times Y_L^{(d)}$ by

$$M_L(A, B) = \mathrm{Prob}(A_{t+1} = B \mid A_t = A) \tag{3.3}$$

where $\mathrm{Prob}(\omega_1|\omega_2)$ denotes the conditional probability of ω_1 given ω_2.

For a finite subset $S \subseteq \Lambda_L^{(d)}$, we define the following function:

$$I_S(\eta, A) = \prod_{x \in A} \eta(x) \prod_{y \in S \setminus A} (1 - \eta(y)) \tag{3.4}$$

for $\eta \in \{0, 1\}^S$ and $A \subseteq S$. It is easy to confirm that M_L is given by the following formula.

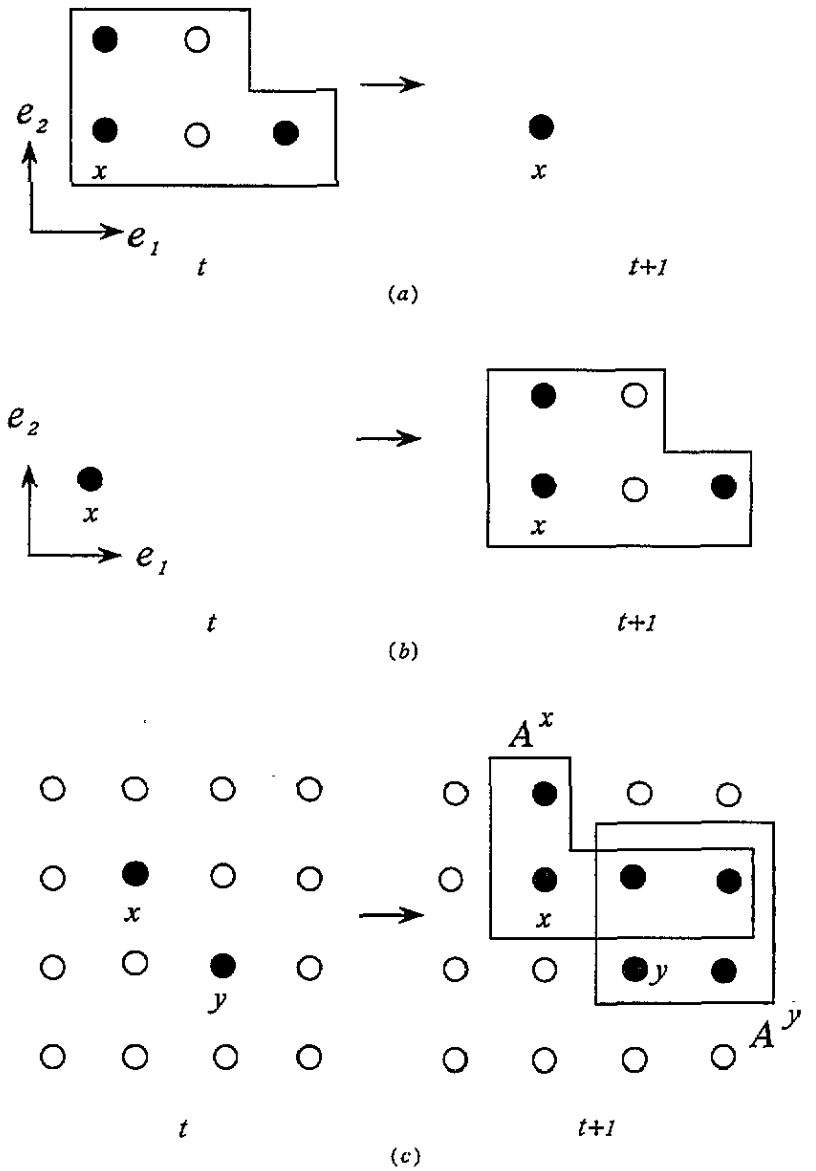

Figure 3. Examples with $R_5^x = \{x, x+e_1, x+2e_1, x+e_2, x+e_1+e_2\}$. (*a*) The creation process: if $\eta_t(y) = 1$ for $y \in A^x = \{x, x+2e_1, x+e_2\}$ and $\eta_t(z) = 0$ for $z \in R_5^x \backslash A^x = \{x+e_1, x+e_1+e_2\}$, $\eta_{t+1}(x) = 1$ with probability $a_{\{0,2e_1,e_2\}}$. (*b*) The branching process: in this case a particle at x branches and sends particles to the sites $x \in A^x = \{x, x+2e_1, x+e_2\}$. This branching occurs with probability $p_{\{0,2e_1,e_2\}}$. (*c*) Particles at x and y branch simultaneously and send particles into A^x and A^y, respectively. The particles are coalescing.

Lemma 3.1. For $\eta_0, \eta_1 \in X_L^{(d)}$, let

$$\tau_x(\{\eta_0(s) : s \in R_n^x\}; \eta_1(x)) = \sum_{A \subseteq R_n^x} I_{R_n^x}(\eta_0, A)[a_A \eta_1(x) + (1 - a_A)(1 - \eta_1(x))] \tag{3.5}$$

and

$$T_L(\eta_0, \eta_1) = \prod_{x \in \Lambda_L^{(d)}} \tau_x(\{\eta_0(s) : s \in R_n^x\}; \eta_1(x)) . \tag{3.6}$$

Then

$$M_L(A, B) = \sum_{\{\eta_0\}} \sum_{\{\eta_1\}} I_{\Lambda_L^{(d)}}(\eta_0, A) T_L(\eta_0, \eta_1) I_{\Lambda_L^{(d)}}(\eta_1, B) \tag{3.7}$$

for $A, B \in Y_L^{(d)}$, where $\sum_{\{\eta\}} \cdot$ denotes the summation over all particle configurations in $X_L^{(d)}$.

3.2. n-range branching process $\tilde{\eta}_t$

Assume that a set of parameters $\{p_A : A \subseteq R_n\}$ is assigned such that

$$0 \leqslant p_A \leqslant 1 \qquad \text{and} \qquad \sum_{A \subseteq R_n} p_A = 1 \tag{3.8}$$

and we define $p_{A^x} = p_A$ for $A^x = A + x$. In the n-range branching process $\tilde{\eta}_t$, a particle at x branches into $|A|$ particles with probability p_A, where $|A|$ is the number of sites in a set A, and these offspring are sent one by one to each site in A^x (see figure 3(b)). If more than one particle are sent to one site, they coalesce into one particle. Therefore, if $\tilde{A}_t = \{x : \tilde{\eta}_t(x) = 1\}$ and each particle at x branches into A^x at time t, then the set $\tilde{A}_{t+1} = \{x : \tilde{\eta}_{t+1}(x) = 1\}$ is given by

$$\tilde{A}_{t+1} = \bigcup_{x \in \tilde{A}_t} A^x \tag{3.9}$$

see figure 3(c). The transition matrix is defined by

$$\tilde{M}_L(A, B) = \text{Prob}(\tilde{A}_{t+1} = B \mid \tilde{A}_t = A) \tag{3.10}$$

for $A, B \in Y_L^{(d)}$.

For $\eta \in X_L^{(d)}$ and $A \in Y_L^{(d)}$, we define

$$J(\eta, A) = \prod_{x \in A} \eta(x) . \tag{3.11}$$

(We assume $J(\eta, \emptyset) = 1$.) Then the transition matrix $\tilde{M}_L$ of the n-range branching process on $\Lambda_L^{(d)}$ is given by the following formula.

Lemma 3.2. For $\eta_0, \eta_1 \in X_L^{(d)}$, let

$$\tilde{\tau}_x(\eta_1(x); \{\eta_0(s) : s \in R_n^x\}) = (1 - \eta_1(x)) + \eta_1(x) \sum_{A \subseteq R_n^x} p_A J(\eta_0, A) \tag{3.12}$$

and

$$\tilde{T}_L(\eta_1, \eta_0) = \prod_{x \in \Lambda_L^{(d)}} \tilde{\tau}_x(\eta_1(x); \{\eta_0(s) : s \in R_n^x\}) . \tag{3.13}$$

Define

$$Q_L(A; \eta_0) = \sum_{\{\eta_1\}} I_{\Lambda_L^{(d)}}(\eta_1, A) \tilde{T}_L(\eta_1, \eta_0) \tag{3.14}$$

for $A \in Y_L^{(d)}, \eta_0 \in X_L^{(d)}$. Then $Q_L(A;\eta_0)$ can be uniquely represented by a linear combination of $\{J(\eta_0, B) : B \in Y_L^{(d)}\}$ and the elements of the transition matrix $\{\tilde{M}_L(A, B)\}$ are given as the coefficients

$$Q_L(A;\eta_0) = \sum_B \tilde{M}_L(A, B) J(\eta_0, B). \tag{3.15}$$

3.3. An example

In order to illustrate the previous formulae, we consider a simple case: $d = 1, n = 2$ and $R_2 = \{0, 1\}$. We put $a_\emptyset = 0$, $a_{\{0\}} = a_{\{1\}} = a$, $a_{\{0,1\}} = b$ and $p_{\{0\}} = p_{\{1\}} = q$, $p_{\{0,1\}} = p$, $p_\emptyset = 1 - p - 2q$, where $0 \leqslant a, b, p, q \leqslant 1$ and $p + 2q \leqslant 1$ are assumed. It is easy to see that the $d = 1$ and $n = 2$ creation process can be identified with the $d = 2$ site (resp. bond) directed percolation with the site (resp. bond) concentration α, if $a = b = \alpha$ (resp. $a = \alpha$ and $b = \alpha(2 - \alpha)$).

Following (3.5) and (3.12), we have

$$\begin{aligned}\tau_x(\{\eta_0(x), \eta_0(x+1)\}; \eta_1(x)) &= (1-\eta_0(x))(1-\eta_0(x+1))(1-\eta_1(x)) \\ &+\eta_0(x)(1-\eta_0(x+1))[a\eta_1(x) + (1-a)(1-\eta_1(x))] \\ &+(1-\eta_0(x))\eta_0(x+1)[a\eta_1(x) + (1-a)(1-\eta_1(x))] \\ &+\eta_0(x)\eta_0(x+1)[b\eta_1(x) + (1-b)(1-\eta_1(x))]\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\tilde{\tau}_x(\eta_1(x); \{\eta_0(x), \eta_0(x+1)\}) &= (1-\eta_1(x)) \\ &+\eta_1(x)\{(1-p-2q) + q\eta_0(x) + q\eta_0(x+1) + p\eta_0(x)\eta_0(x+1)\}.\end{aligned} \tag{3.17}$$

When $L = 2$, $\Lambda_2^{(1)} = \{1, 2\}$ and $T_2(\eta_0, \eta_1) = \tau_1 \times \tau_2$, $\tilde{T}_2(\eta_1, \eta_0) = \tilde{\tau}_1 \times \tilde{\tau}_2$. Lemmas 3.1 and 3.2 give the following:

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ (1-a)^2 & (1-a)a & (1-a)a & a^2 \\ (1-a)^2 & (1-a)a & (1-a)a & a^2 \\ (1-b)^2 & (1-b)b & (1-b)b & b^2 \end{pmatrix} \tag{3.18}$$

and

$$\tilde{M}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1-p-2q & q & q & p \\ 1-p-2q & q & q & p \\ (1-p-2q)^2 & q(2-2p-3q) & q(2-2p-3q) & 2p-p^2+2q^2 \end{pmatrix}. \tag{3.19}$$

If we put $a = b = p$ and $q = 0$, these matrices become (2.4) and (2.5).

3.4. Duality relation

In this subsection we prove the following proposition.

Proposition 3.3. If

$$a_A = \begin{cases} \displaystyle\sum_{C: C \subseteq R_n, C \cap A \neq \emptyset} p_C & \text{if} \quad A \neq \emptyset \\ 0 & \text{if} \quad A = \emptyset \end{cases} \tag{3.20}$$

then the equalities

$$\sum_{C:C\cap B\neq\emptyset} M_L(A,C) = \sum_{C:C\cap A\neq\emptyset} \tilde{M}_L(B,C) \tag{3.21}$$

hold for all $A, B \in Y_L^{(d)}$ with any $L \geqslant n$.

Proof. By the definitions of the transition matrices, (3.3) and (3.10),

$$\sum_C M_L(A,C) = 1 \qquad \text{and} \qquad \sum_C \tilde{M}_L(A,C) = 1\,. \tag{3.22}$$

The equality (3.21) is thus equivalent to

$$\sum_{C:C\cap B=\emptyset} M_L(A,C) = \sum_{C:C\cap A=\emptyset} \tilde{M}_L(B,C)\,. \tag{3.23}$$

The formula (3.7) of lemma 3.1 gives

$$\sum_{C:C\cap B=\emptyset} M_L(A,C) = \sum_{\{\eta_0\}}\sum_{\{\eta_1\}} I_{\Lambda_L^{(\omega)}}(\eta_0,A)T_L(\eta_0,\eta_1)\prod_{x\in B}(1-\eta_1(x)) \tag{3.24}$$

since the identity

$$\sum_{C:C\cap B=\emptyset} I_{\Lambda_L^{(\omega)}}(\eta,C) = \prod_{x\in B}(1-\eta(x)) \tag{3.25}$$

holds for any $\eta \in X_L^{(d)}$ and $B \in Y_L^{(d)}$. On the other hand, we have

$$\sum_{C:C\cap A=\emptyset} \tilde{M}_L(B,C) = \sum_{\{\eta_0\}}\sum_C \tilde{M}_L(B,C)J(\eta_0,C)I_{\Lambda_L^{(\omega)}}(\eta_0,\Lambda_L^{(d)}\setminus A) \tag{3.26}$$

since

$$\sum_{\{\eta_0\}} J(\eta_0,B)I_{\Lambda_L^{(\omega)}}(\eta_0,\Lambda_L^{(d)}\setminus A) = \begin{cases} 1 & \text{if} \quad A\cap B=\emptyset \\ 0 & \text{otherwise} \end{cases} \tag{3.27}$$

for any $A, B \in Y_L^{(d)}$. The formula (3.15) of lemma 3.2 gives

$$\text{RHS of (3.26)} = \sum_{\{\eta_0\}} Q_L(B;\eta_0)I_{\Lambda_L^{(\omega)}}\big(\eta_0,\Lambda_L^{(d)}\setminus A\big)\,. \tag{3.28}$$

Therefore, by the definition (3.14),

$$\begin{aligned}\sum_{C:C\cap A=\emptyset} \tilde{M}_L(B,C) &= \sum_{\{\eta_0\}}\sum_{\{\eta_1\}} I_{\Lambda_L^{(\omega)}}(\eta_1,B)\tilde{T}_L(\eta_1,\eta_0)I_{\Lambda_L^{(\omega)}}\big(\eta_0,\Lambda_L^{(d)}\setminus A\big) \\ &= \sum_{\{\eta_0\}}\sum_{\{\eta_1\}} I_{\Lambda_L^{\omega}}(\eta_0,A)\tilde{T}_L(1-\eta_1,1-\eta_0)\prod_{x\in B}(1-\eta_1(x))\prod_{y\in\Lambda_L^{(\omega)}\setminus B}\eta_1(y)\,.\end{aligned} \tag{3.29}$$

Here we have used the identity

$$I_{\Lambda_L^{(\omega)}}\big(1-\eta_0,\Lambda_L^{(d)}\setminus A\big) = I_{\Lambda_L^{(\omega)}}(\eta_0,A)\,. \tag{3.30}$$

Comparing (3.24) and (3.29), we find that (3.23) follows if

$$\ell(B,\eta_0) = r(B,\eta_0) \tag{3.31}$$

for any $B \in Y_L^{(d)}$, $\eta_0 \in X_L^{(d)}$, where

$$\ell(B,\eta_0) \equiv \sum_{\{\eta_1\}} T_L(\eta_0,\eta_1)\prod_{x\in B}(1-\eta_1(x)) \tag{3.32}$$

and

$$r(B,\eta_0) \equiv \sum_{\{\eta_1\}} \tilde{T}_L(1-\eta_1, 1-\eta_0) \prod_{x\in B}(1-\eta_1(x)) \prod_{y\in\Lambda_L^{(d)}\setminus B} \eta_1(y)\,. \tag{3.33}$$

Since $T_L(\eta_0,\eta_1)$ and $\tilde{T}_L(\eta_0,\eta_1)$ are defined by the products of τ_x and $\tilde{\tau}_x$ as (3.6) and (3.13), respectively, simple calculation gives

$$\ell(B,\eta_0) = \prod_{z\in B}\left\{\sum_{A\subseteq R_n^z}(1-a_A) I_{R_n^z}(\eta_0, A)\right\} \tag{3.34}$$

$$r(B,\eta_0) = \prod_{z\in B}\left\{\sum_{A\subseteq R_n^z} p_A J(1-\eta_0, A)\right\}. \tag{3.35}$$

Using an identity similar to (3.25), we have

$$\begin{aligned}\sum_{A\subseteq R_n^z} p_A J(1-\eta_0, A) &= \sum_{A\subseteq R_n^z} p_A \sum_{C:C\subseteq R_n^z, C\cap A=\emptyset} I_{R_n^z}(\eta_0, C) \\ &= \sum_{C\subseteq R_n^z}\left\{\sum_{A:A\subseteq R_n^z, A\cap C=\emptyset} p_A\right\} I_{R_n^z}(\eta, C)\,. \end{aligned}\tag{3.36}$$

Thus (3.31) is satisfied if

$$1-a_A = \sum_{C:C\subseteq R_n, C\cap A=\emptyset} p_C \tag{3.37}$$

for any $A \subseteq R_n$. It is easy to see that (3.37) is equivalent to the condition (3.20), because of (3.8). □

For the example in section 3.3, the condition (3.20) is given as $a = p+q$ and $b = p+2q$. As a special case it is satisfied if $a = b = p$ and $q = 0$ as we have shown in section 2.

Remark 3.1. The interacting particle systems are constructed by a method called the *graphical representation*, where we consider the generalized directed percolation problems on the spatio-temporal hyperplane. We can represent the n-range creation process η_t and the n-range branching process $\tilde{\eta}_t$ using the common gadgets on the same hyperplane $\Lambda_L^{(d)} \times \{0, 1, 2, \ldots\}$. Since these two processes can be defined on a common probability space, we can find that the following relation holds between the respective events with probability 1:

$$\{A_t \cap B \neq \emptyset\} = \{A \cap \tilde{B}_t \neq \emptyset\} \tag{3.38}$$

for any $A, B \in Y_L^{(d)}$ and $t \in \{0, 1, 2, \ldots\}$, where $A_t = \{x : \eta_t(x) = 1\}$ and $\tilde{B}_t = \{x : \tilde{\eta}_t(x) = 1\}$ with $A_0 = A$ and $\tilde{B}_0 = B$. By equating the probabilities of these two events, we have

$$\text{Prob}(A_t \cap B \neq \emptyset | A_0 = A) = \text{Prob}(A \cap \tilde{B}_t \neq \emptyset | \tilde{B}_0 = B) \tag{3.39}$$

which is called the *coalescing duality relation*. For more detail, see section 5b of Durrett [3]. It should be noted that the relation (3.21) is nothing but (3.39) with $t = 1$. Constructing two or more stochastic processes on a common probability space and making comparisons among them is generally called the *coupling technique* and is well known as a useful tool in probability theory [2]. In this subsection, however, we have used the basic properties of the transition matrices of η_t and $\tilde{\eta}_t$ and showed another derivation of the relation (3.21) for making the present paper self-contained.

Remark 3.2. The reader should be reminded that the 'duality' relation discussed by Kinzel [31] regarding a mapping of rules of cellular automata is different from the coalescing duality relation discussed in this paper. Dhar *et al* [33] introduced another duality transformation to relate the two-dimensional bond directed percolation with some random resistance problem. This duality shall be called the *planar lattice duality* and is also different from the coalescing duality. It is possible to extend Dhar's argument [33, 34] for the present n-range creation process, which contains the bond directed percolation as a special case. More detail will be reported elsewhere in the near future.

4. Equality of the characteristic polynomials

For a finite matrix M the characteristic polynomial is defined by

$$\gamma_M(\lambda) = \det(M - \lambda E) \tag{4.1}$$

where E is the identity matrix. Since the eigenvalues of M are given by the roots of the equation $\gamma_M(\lambda) = 0$, the equality $\gamma_{M_1}(\lambda) = \gamma_{M_2}(\lambda)$ means the equality of all eigenvalues. For a given $\Lambda_L^{(d)}$, we define the $2^{L^d} \times 2^{L^d}$ matrices on $Y_L^{(d)} \times Y_L^{(d)}$ as

$$V_+ \equiv (V_+(A, B)) = \begin{cases} 1 & \text{if} \quad A \subseteq B \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

and

$$V_- \equiv (V_-(A, B)) = \begin{cases} 1 & \text{if} \quad A \supseteq B \\ 0 & \text{otherwise}. \end{cases} \tag{4.3}$$

Since we can make V_+ and V_- upper and lower triangular matrices with all diagonal elements unity,

$$\det(V_+) = \det(V_-) = 1. \tag{4.4}$$

In this section we will prove the following main theorem of the present paper.

Theorem 4.1. Consider the n-range creation process η_t with parameters $\{a_A : A \in R_n\}$ and the n-range branching process with parameters $\{p_A : A \in R_n\}$.

(i) Let M_L and $\tilde{M}_L$ be the transition matrices corresponding to η_t and $\tilde{\eta}_t$ defined on a finite lattice $\Lambda_L^{(d)}$, respectively. If

$$a_A = \begin{cases} \displaystyle\sum_{C : C \subseteq R_n, C \cap A \neq \emptyset} p_C & \text{if} \quad A \neq \emptyset \\ 0 & \text{if} \quad A = \emptyset \end{cases} \tag{4.5}$$

then the equality

$$\gamma_{M_L}(\lambda) = \gamma_{\tilde{M}_L}(\lambda) \tag{4.6}$$

holds for any $L \geqslant n$.

(ii) Let ϕ_λ and φ_λ be the left and the right eigenvectors of M_L with an eigenvalue λ, respectively. If (4.5) holds, then the corresponding left and the right eigenvectors of $\tilde{M}_L$ with the same eigenvalue λ are given by $\varphi_\lambda^t V_-^{-1} V_+^{-1}$ and $V_+ V_- \phi_\lambda^t$, respectively, where φ_λ^t (resp. ϕ_λ^t) denotes the transposed vector of φ_λ (resp. ϕ_λ).

We note in passing that the transpose of V_+ is equal to V_-. Therefore proposition 3.3 and the following proposition give theorem 4.1, where $\tilde{M}_L^t$ denotes the transposed matrix of $\tilde{M}_L$.

Proposition 4.2. If the duality relation (3.21) holds, M_L and $\tilde{M}_L^{\dagger}$ are conjugate. That is,

$$\tilde{M}_L^{\dagger} = W^{-1} M_L W \tag{4.7}$$

with $W = V_+ V_-$.

Proof of Proposition 4.2. It is noted that the relation (3.21) is rewritten as

$$\sum_{D:D\subseteq B} \sum_{C:D\subseteq C} M_L(A, C) = \sum_{D:D\subseteq A} \sum_{C:D\subseteq C} \tilde{M}_L(B, C) \tag{4.8}$$

or equivalently

$$\sum_D \sum_C 1_{\{D\subseteq B\}} 1_{\{D\subseteq C\}} M_L(A, C) = \sum_D \sum_C 1_{\{D\subseteq A\}} 1_{\{D\subseteq C\}} \tilde{M}_L(B, C). \tag{4.9}$$

by using the indicator function

$$1_{\{\omega\}} = \begin{cases} 1 & \text{if } \omega \text{ is satisfied} \\ 0 & \text{otherwise.} \end{cases} \tag{4.10}$$

By the definitions (4.2) and (4.3), (4.7) follows (4.9). □

5. Concluding remarks

In this paper we have introduced two kinds of discrete-time non-equilibrium lattice models, the n-range creation process η_t and the n-range branching process η_t, which are parametrized by $\{a_A\}$ and $\{p_A\}$, respectively. Theorem 4.1 implies that if the process $\tilde{\eta}_t$ shows a continuous phase transition at $p_A = p_{Ac}$ for $A \subseteq R_n$ and critical phenomena are observed, then the process η_t also shows the phase transition and critical phenomena at $a_A = a_{Ac}$ which is related with p_{Ac} by (4.5) and that these two critical phenomena belong to the same universality class.

The existence of phase transitions are guaranteed for the present processes by the following argument. First we consider the simplest case, $d = 1$ and $n = 2$ branching process, given in section 3.3. If we assume that there is only one particle at time 0, the expectation of particle number at time $t = 1$ is $\langle |\eta_1| \rangle = 2p + 2q$, which is less than 1 for $p < \frac{1}{2} - q$. It is easy to prove that the trivial absorbing state is the unique stationary state for any initial states if $p < \frac{1}{2} - q$. On the other hand, by the contour method (i.e. Peierls' argument, see, for example, Durrett [3]) we can prove that there is another active stationary state if $p > \frac{80}{81}$. Therefore there should be a critical line between these two regions on a (p, q)-plane. Quite recently Liggett [35] proved the following remarkable results for the $d = 1, n = 2$ creation process with $b \geqslant a$. (i) If $b < 2(1 - a)$, the trivial absorbing state is the unique stationary state. (ii) If $\frac{1}{2} < a \leqslant 1$ and $b \geqslant 4a(1 - a)$, there is another active stationary state. For more details on the phase diagrams of these models, see [3, 31]. Next we consider general cases with $n \geqslant 3$. We find processes die out with probability 1 if $\sum_{A:A\subseteq R_n, A\neq\emptyset} |A| p_A < 1$, where $|A|$ denotes the number of sites in a set A. If we choose parameters so that $p_{\{0,1\}} > \frac{80}{81}$, there is a positive probability to have the active stationary states. Then there should be a critical surface between these two regions in the 2^n-dimensional parameter space.

It should be remarked that theorem 4.1 will be generalized to other non-equilibrium lattice models which have coalescing dual processes. However, the generalization to the models which have no coalescing dual processes [36], the annihilation-type models such as a branching annihilating random walk [6, 14–17] and multi-species models is not trivial and should be studied in the future.

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References

[1] Harris T E 1974 Contact interactions on a lattice *Ann. Prob.* **2** 969–88
[2] Liggett T M 1985 *Interacting Particle Systems* (New York: Springer)
[3] Durrett R 1988 *Lecture Notes on Particle Systems and Percolation* (California: Wadsworth and Brooks/Cole)
[4] Dickman R and Burschka M A 1988 Nonequilibrium critical poisoning in a single-species model *Phys. Lett.* **127A** 132–7
[5] Ferreira A L C and Mendiratta S K 1993 Mean-field approximation with coherent anomaly method for a non-equilibrium model *J. Phys. A: Math. Gen.* **26** L145–50
[6] Katori M and Konno N 1993 Phase transitions in contact process and its related processes *Formation, Dynamics and Statistics of Patterns* vol 2 ed K Kawasaki and M Suzuki (Singapore: World Scientific) pp 23–72
[7] Katori M and Konno N 1993 Bounds for the critical line of the θ-contact processes with $1 \leqslant \theta \leqslant 2$ *J. Phys. A: Math. Gen.* **26** 6597–614
[8] Jensen I and Dickman R 1994 Series analysis of the generalized contact process *Physica* **203A** 175–88
[9] Jensen I and Dickman R 1993 Time-dependent perturbation theory for diffusive non-equilibrium lattice models *J. Phys. A: Math.Gen.* **26** L151–7
[10] Matsuda H, Ogita N, Sasaki A and Satō K 1992 Statistical mechanics of population—the lattice Lotka-Volterra model *Prog. Theor. Phys.* **88** 1035–49
[11] Konno N 1995 Asymptotic behaviour of basic contact process with rapid stirring *J. Theor. Probab.* to appear
[12] Konno N 1994 *Phase Transitions of Interacting Particle Systems* (Singapore: World Scientific)
[13] Katori M 1994 Rigorous results for the diffusive contact processes in $d \geqslant 3$ *J. Phys. A: Math. Gen.* **27** 7327–41
[14] Bramson M and Gray L 1985 The survival of branching annihilating random walk *Z. Wahr. Verw. Geb.* **68** 447–60
[15] Takayasu H and Tretyakov A Yu 1992 Extinction, survival, and dynamical phase transition of branching annihilating random walk *Phys. Rev. Lett.* **68** 3060–3
[16] Jensen I 1993 Critical behaviour of branching annihilating random walks with an odd number of offsprings *Phys. Rev.* E **47** R1–4
[17] Inui N 1993 Application of the coherent anomaly method to the branching annihilation random walk *Phys. Lett.* **184A** 79–82
[18] Dickman R 1989 Universality and diffusion in non-equilibrium critical phenomena *Phys. Rev.* B **40** 7005–10
[19] Dickman R 1990 Nonequilibrium critical behaviour of the triplet annihilation model *Phys. Rev.* A **42** 6985–90
[20] Katori M and Konno N 1992 On the extinction of Dickman's reaction-diffusion processes *Physica* **186A** 578–90
[21] Ziff R M, Gulari E and Barshad Y 1986 Kinetic phase transitions in an irreversible surface-reaction model *Phys. Rev. Lett.* **56** 2553–6
[22] Grassberger P and de la Torre A 1979 Reggeon field theory (Schlögl's first model) on a lattice: Monte Carlo calculations of critical behaviour *Ann. Phys.* **122** 373–96
[23] Dickman R and Jensen I 1991 Time-dependent perturbation theory for non-equilibrium lattice models *Phys. Rev. Lett.* **67** 2391–4
[24] Cardy J L and Sugar R L 1980 Directed percolation and Reggeon field theory *J. Phys. A: Math. Gen.* **13** L423–7
[25] Janssen H K 1981 On the non-equilibrium phase transition in reaction-diffusion systems with an absorbing stationary state *Z. Phys.* B **42** 151–4
[26] Grassberger P 1982 On phase transitions in Schlögl's second model *Z. Phys.* B **47** 365–74

[27] Dickman R 1993 Nonequilibrium phase transitions in catalysis and population models *Int. J. Mod. Phys.* C 4 271–7
[28] Jensen I and Dickman R 1993 Nonequilibrium phase transitions in systems with infinitely many absorbing states *Phys. Rev.* E **48** 1710–25
[29] Mendes J F F, Dickman R, Henkel M and Marques M C 1994 Generalized scaling for models with multiple absorbing states *J. Phys. A: Math. Gen.* **27** 3019–28
[30] Kim M H, Yoon D H and Kim C H 1994 Read at the *Autumn Meeting of the Phys. Soc. of Japan* Shizuoka University
[31] Kinzel W 1985 Phase transitions of cellular automata *Z. Phys.* B 58 229–44
[32] ben-Avraham D, Bidaux R and Schulman L S 1991 Transfer-matrix methods and results for directed percolation *Phys. Rev.* A **43** 7093–6
[33] Dhar D, Barma M and Phani M K 1981 Duality transformation for two-dimensional directed percolation and resistance problems *Phys. Rev. Lett.* **47** 1238–41
[34] Dhar D 1982 Diode-resistor percolation in two and three dimensions: I. Upper bounds on critical probability *J. Phys. A: Math. Gen.* **15** 1849–58
[35] Liggett T M 1994 Survival of discrete time growth models, with applications to oriented percolation *Preprint*
[36] Katori M 1994 Reformulation of Gray's duality for attractive spin systems and its applications *J. Phys. A: Math. Gen.* **27** 3191–211